April 3: Krull Domains and the Mori-Nagata Theorem, part 3

The Mori-Nagata Theorem

We now turn our attention to the Mori-Nagata Theorem. The proof is based upon ideas of Nagata, Rees, Querre, and McAdam. The original proof due to Nagata used the Cohen structure theorem and properties of completions; in particular, it used the fact that a complete local domain is finite over a complete regular local ring. More modern treatments, like the one below, avoid the use of completions.

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Lemma K2. Let R be a Noetherian domain with integral closure S. Take $a \in R$, assume that $Q \subseteq S$ is minimal prime over $(aS :_S b)$ and set $P := Q \cap R$. Then P is an associated prime of R/aR.

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If we do this for the finitely many generators of P, it follows that there exists $s \in S \backslash Q$, $t \geq 1$ and a ring $R \subseteq R_0 \subseteq S$, such that $P^t \cdot sb \subseteq aR_0$ and R_0 is a finite R-module,

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Thus, for some n, $cs^n b^n \notin a^n R$. Therefore, P^{nt} consists of zero divisors modulo $a^n R$.

Since P is maximal, it follows that $P \in Ass(R/a^n R)$, and hence $P \in Ass(R/aR)$, which gives what we want. □

The Mori-Nagata Theorem, continued

Lemma L2. Let R be a Noetherian domain and set $A := S \cap T$, where T is the global transform of R and S is the integral closure of R. If $P \subseteq A$ is a maximal ideal and P is an associated prime of a principal ideal, then A_P is a DVR.

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Proof. Suppose $P = (aA : b)$ is maximal. If $m := R \cap P$, then m is maximal (since A is integral over R) and $m(b/a)$ is contained in $A \subseteq T$, so $Jm(b/a) \subseteq R$, for J a product of maximal ideals.

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Thus, P is invertible, so P_P is principal. i.e., A_P is a DVR. □

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Moreover, since each contraction to A is minimal over PA (by lying over), these contractions are finite in number since A is Noetherian.

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Thus, only finitely many height one primes in S contract to P. □

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Thus, S is the integral closure of R_0. Changing notation, we may start again, assuming simply that S is the integral closure of R.

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Since R/aR has only finitely many associated primes, there can only be finitely many height one primes containing aS.

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Finally, suppose that $x \in \bigcap S_Q$, where the intersection ranges over the height one primes of S.

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But, $k(Q/Q') = k(Q)$ and $k(P/P') = k(P)$, so, $[k(Q) : k(P)] < \infty$.

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Thus, by induction on the height of P, in the integral closure of R/P' in $k(Q')$, there are only finitely many primes lying over P/P'. Therefore, S/Q' contains only finitely many primes lying over P/P'.

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Thus, there are only finitely many primes in S containing Q' lying over P. Since this holds for each of the finitely many minimal primes of aS, we conclude that there are also only finitely many primes of height greater than one in S lying over P.

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This completes the proof of the Mori-Nagata theorem.

Two applications of the Mori-Nagata Theorem

As applications of the Mori-Nagata theorem, we will prove that the integral closure of a two-dimensional Noetherian domain is Noetherian and that a complete local domain is satisfies N_2.

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Proof. By the Mori-Nagata theorem, S is a Krull domain, so by Nishimura's theorem (Theorem 2C), it suffices to prove that S/Q is Noetherian for all height one primes $Q \subseteq S$.

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Proof. By the Mori-Nagata theorem, S is a Krull domain, so by Nishimura's theorem (Theorem 2C), it suffices to prove that S/Q is Noetherian for all height one primes $Q \subseteq S$.

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